

Math 213 – Elementary Linear Algebra

Section 5.2: Orthogonal Complements and Orthogonal Projections

From orthogonal vectors to orthogonal spaces

Last Time

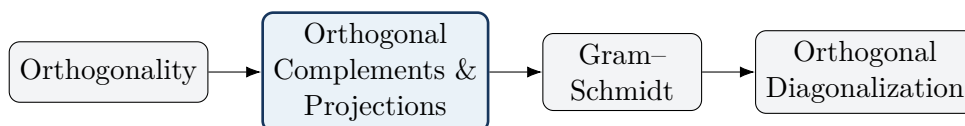
Last time we learned that orthogonal bases allow us to find coordinates using dot products instead of solving systems of equations.

Today

Today we continue exploring the power of orthogonality by studying orthogonal complements and orthogonal projections. These ideas allow us to split any vector into a component that lies in a subspace and a component that is perpendicular to it.

By the end of today's lesson you should be able to:

- Find the orthogonal complement of a subspace.
- Find a basis for the orthogonal complement.
- Use $\text{Null}(A)$ and $\text{Null}(A^T)$ to find orthogonal complements.
- Compute the orthogonal projection of a vector onto a subspace.
- Find the orthogonal decomposition of a vector.



Motivation

Think-Pair-Share

Let $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Calculate:

$$\mathbf{v} \cdot \mathbf{w}_1 =$$

$$\mathbf{v} \cdot \mathbf{w}_2 =$$

$$\mathbf{w}_1 \cdot \mathbf{w}_2 =$$

Is $\{\mathbf{w}_1, \mathbf{w}_2\}$ an orthogonal set? Which set is orthogonal?

Orthogonal Complements

Definition

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is **orthogonal to a subspace** W of $\mathbb{R}^n$ if $\mathbf{v}$ is orthogonal to every vector in W , that is, $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w}$ in W .

The **orthogonal complement** of W is

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for every } \mathbf{w} \in W\}.$$

In other words,

$$W^\perp = \{\text{all vectors perpendicular to } W\}.$$

Example

Let $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\{\mathbf{v} \text{ in } \mathbb{R}^n \text{ such that } \mathbf{v} \cdot \mathbf{w} = 0\} =$$

$$\{\mathbf{v} \text{ in } \mathbb{R}^n \text{ such that } \mathbf{v} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{x} \text{ in } \text{Span}(\mathbf{w})\} =$$

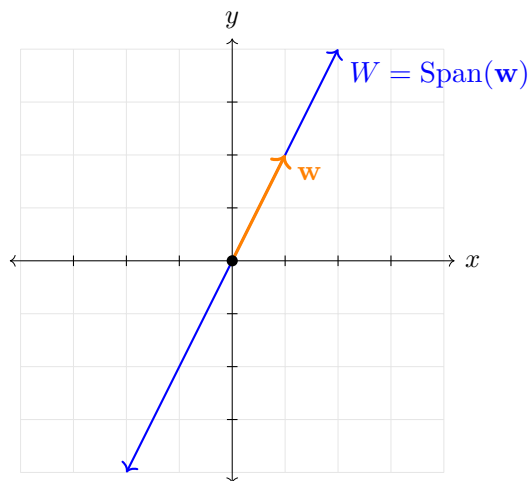
Example

Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x - y = 0 \right\}$. Find $W^\perp$ and a basis for $W^\perp$.

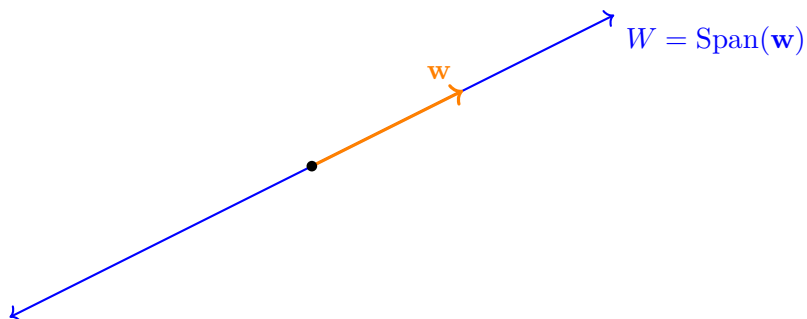
Example

Suppose W is the objects below. Describe $W^\perp$ for each object.

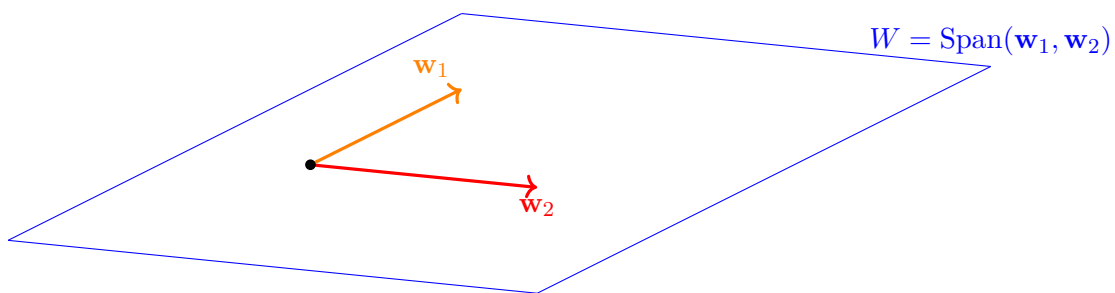
1. A line in $\mathbb{R}^2$.



2. A line in $\mathbb{R}^3$.



3. A plane in $\mathbb{R}^3$.



Theorem

Let W be a subspace of $\mathbb{R}^n$. Then

1. $W^\perp$ is also a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $W \cap W^\perp = \{\mathbf{0}\}$, meaning the only vector in both W and $W^\perp$ is $\mathbf{0}$
4. If $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then $\mathbf{v}$ is in $W^\perp$ if and only if $\mathbf{v}$ is orthogonal to each $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Finding a Basis for the Orthogonal Complement**Theorem (Fundamental Subspaces Theorem)**

For any $m \times n$ matrix A ,

$$\text{Row}(A)^\perp = \text{Null}(A), \quad \text{Col}(A)^\perp = \text{Null}(A^T).$$

We call $\text{Row}(A)$, $\text{Null}(A)$, $\text{Col}(A)$, and $\text{Null}(A^T)$ the **four fundamental subspaces** of A .

Think

Why does this make sense?

Strategy**Finding a Basis for $W^\perp$**

If $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then

1. Put the spanning vectors into a matrix as columns.
2. Compute

$$W^\perp = \text{Col}(A)^\perp = \text{Null}(A^T).$$

3. A basis for $\text{Null}(A^T)$ is a basis for $W^\perp$.

(or if they are row vectors, put them as rows of a matrix A , and $W^\perp = \text{Null}(A)$)

Example

Let

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}.$$

Find a basis for $W^\perp$.

Orthogonal Projection

Think–Pair–Share

Now suppose we have a vector $\mathbf{y}$ that is not in a subspace W .
How can we find the vector in W that is “closest” to $\mathbf{y}$?

Definition

Let W be a subspace of $\mathbb{R}^n$ and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be an *orthogonal basis* for W . For any vector $\mathbf{v}$ in $\mathbb{R}^n$, the **orthogonal projection of $\mathbf{v}$ onto W** is defined as

$$\text{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 + \cdots + \left(\frac{\mathbf{v} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k} \right) \mathbf{u}_k.$$

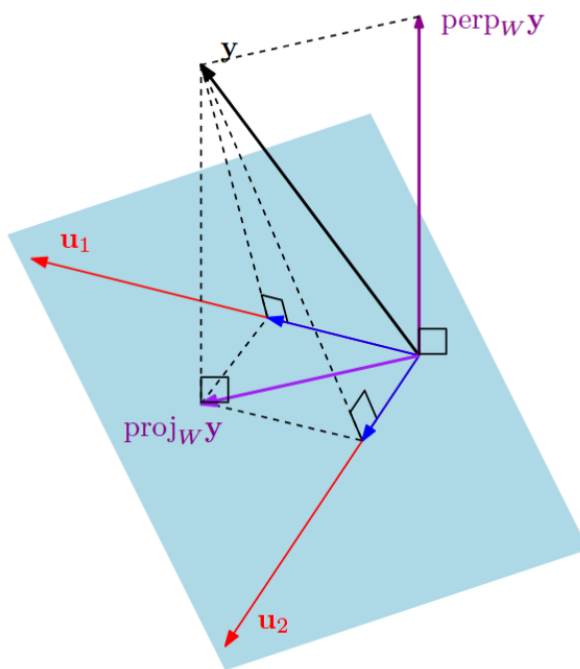
The **component of $\mathbf{v}$ orthogonal to W** is the vector

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

WARNING! This definition requires the basis to be **orthogonal**!

Big Idea

BONUS: If the basis is also **orthonormal**...



Big Idea

For any vector $\mathbf{v}$, $\text{proj}_W(\mathbf{v}) \in W$ and $\text{perp}_W(\mathbf{v}) \in W^\perp$.

Orthogonal Decomposition

Theorem (Orthogonal Decomposition Theorem)

Let W be a subspace of $\mathbb{R}^n$. Then every vector $\mathbf{v} \in \mathbb{R}^n$ can be uniquely written as

$$\mathbf{v} = \text{proj}_W(\mathbf{v}) + \text{perp}_W(\mathbf{v}).$$

Why?

Every vector can be uniquely decomposed into

$$\text{Part in } W + \text{Part in } W^\perp.$$

Example

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}$. Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Find the orthogonal decomposition of $\mathbf{v}$ with respect to W . This means find:

1. $\text{proj}_W(\mathbf{v})$
2. $\text{perp}_W(\mathbf{v})$

Theorem

If W is a subspace of $\mathbb{R}^n$, then

$$\dim W + \dim W^\perp = n.$$

Why would this be true?

Reflection

Today we learned about orthogonal complements, projections, and orthogonal decompositions. In your own words, explain how these three ideas are connected.

Section 5.2 Summary

Today we discussed, how to:

- Find the orthogonal complement of a subspace W .
- Find a basis for $W^\perp$ using $\text{Null}(A)$ or $\text{Null}(A^T)$.
- Compute orthogonal projections onto a subspace.
- Find the perpendicular component $\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$.
- Write the orthogonal decomposition $\mathbf{v} = \text{proj}_W(\mathbf{v}) + \text{perp}_W(\mathbf{v})$.

Next Time

We now have several results that require an orthogonal basis. But what if we don't have any orthogonal basis? Next we discuss how to obtain an orthogonal basis for a subspace.